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# Prime spectra of additive categories (I)

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**Abstract**

In this paper, by introducing appropriate notions of multiplication of ideals (resp. sieves) of an additive category $\mathcal{C}$, we associate to the category a topological space Spec($\mathcal{C}$) consisting of all globally prime ideals. We can also associate to each object $C \in \mathcal{C}$ a space Spec($C$) consisting of all prime 2-sided sieves at $C$ in such a way that each local spectrum Spec($C$) is a retract of the global spectrum Spec($\mathcal{C}$). © 1997 Elsevier Science B.V.

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## 0. Introduction

In Algebraic Geometry one studies topological spaces $X$ that are locally like the prime spectrum of a ring, i.e. schemes (e.g. see [5]). In this paper, by introducing appropriate notions of multiplication of ideals (resp. sieves) of an additive category $\mathcal{C}$, and of primeness with respect to the multiplication, we can associate to the category a topological space Spec($\mathcal{C}$) consisting of all globally prime ideals. We can also associate to each object $C \in \mathcal{C}$ a space Spec($C$) consisting of all prime two-sided sieves at $C$ in such a way that each local spectrum Spec($C$) is a retract of the global spectrum Spec($\mathcal{C}$).

We begin by looking at sieves at an object in $\mathcal{C}$ and by defining prime sieves torsion theoretically, then we shall see that this notion leads naturally to a notion of product of sieves. Product of sieves can be used to define a stronger notion of primeness in sieves. The product of sieves naturally induces a product of ideals (subfunctors of Hom$(-,-)$). Thus we can define prime ideals.

Sections 3, 4, 6 and 7 by using our notions of product and prime, we will many deep results in noncommutative rings, can be extended to additive also suggests that our definition may be in some sense reasonable.

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Some examples of such results are: Krull's Separation Lemma, characterizations of Jacobson radical, the result that right Artinian rings are right Noetherian; and the Wedderburn–Artin Structure Theorem. The work in this paper has been inspired by the work done by Street [7] and Kelly [3].

## 1. Preliminaries

Throughout we let $C$ denote an additive category. Let **mod**-$C = [C^{\mathrm{op}}, \mathbf{Ab}]_{\mathrm{Additive}}$. We will call this functor category *the category of right $C$-modules* and the objects of this category will be called *right $C$-modules*. We shall think of our additive category $C$ as a ring with many objects, and hence we will assume that $C$ is small throughout the remainder of this paper.

**Definition 1.1.** Define a *torsion theory* $(\mathfrak{T}, \mathfrak{F})$ on **mod**-$C$ to be a pair of classes of right $C$-modules satisfying:

(i) for all $M \in \mathfrak{T}$ and for all $N \in \mathfrak{F}$ we have

$$\mathrm{Hom}(M,N) = \{0\},$$

(ii) if $M \in$ **mod**-$C$ is such that

$$\mathrm{Hom}(M,N) = \{0\}$$

for all $N \in \mathfrak{F}$ then $M \in \mathfrak{T}$,

(iii) if $N \in$ **mod**-$C$ is such that

$$\mathrm{Hom}(M,N) = \{0\}$$

for all $M \in \mathfrak{T}$ then $N \in \mathfrak{F}$.

If $(\mathfrak{T}, \mathfrak{F})$ is a torsion theory then the class $\mathfrak{T}$ is called a *torsion class* and the class $\mathfrak{F}$ is called a *torsion free class*.

A torsion theory $(\mathfrak{T}, \mathfrak{F})$ is called *hereditary* if $\mathfrak{T}$ is closed under submodules.

**Definition 1.2.** Let $\mathcal{J}$ be an additive Grothendieck topology on $C$. A right $C$-module $F: C^{\mathrm{op}} \to \mathbf{Ab}$ is called *$\mathcal{J}$-torsion* if for each $C \in C$ and each $x \in F(C)$ there is a sieve $S_x \in \mathcal{J}(C)$ such that $F(f)(x) = 0$ for all $f \in S_x$.

**Lemma 1.3.** *Let $\mathcal{J}$ be an additive Grothendieck topology on $C$. If we define*

$$\mathfrak{T}_{\mathcal{J}} = \{M \in \mathbf{mod}\text{–}C \mid M \text{ is } \mathcal{J}\text{–torsion}\},$$

*then $\mathfrak{T}_{\mathcal{J}}$ is a hereditary torsion class.*

**Lemma 1.4.** *Let $(\mathfrak{T}, \mathfrak{F})$ be a hereditary torsion theory on* **mod**-$C$. *The assignment*

$$\mathcal{J}_{\mathfrak{T}}(C) = \{i: I \hookrightarrow \mathrm{Hom}(-, C) \mid \mathrm{Cok}(i) \in \mathfrak{T}\}$$

*for each $C \in C$ is an additive Grothendieck topology.*

**Proposition 1.5.** *For an additive category $\mathcal{C}$, there exist bijections between*
(a) *Additive Grothendieck topologies on $\mathcal{C}$*
(b) *Herediatary torsion theories on* **mod**-$\mathcal{C}$.

**Proof.** This proposition follows from the preceding two lemma's. $\square$

**Lemma 1.6.** *Let $\mathcal{J}$ be an additive Grothendieck topology on $\mathcal{C}$. If $(\mathfrak{T}, \mathfrak{F})$ denotes the hereditary torsion theory corresponding to $\mathcal{J}$ then $M \in \mathfrak{F}$ if and only if $M$ satisfies the following condition:*
(TF) *Whenever there is a $C \in \mathcal{C}$ and an $x \in F(C)$ such that there exists a sieve $S_x \in \mathcal{J}(C)$ such that $F(f)(x) = 0$ for all $f \in S_x$ then $x = 0$.*

**Proof.** ($\Rightarrow$) Suppose that $M \in \mathfrak{F}$. Let $C \in \mathcal{C}$ and let $x \in M(C)$ be such that there exist $S \in \mathcal{J}(C)$ with $M(f)(x) = 0$ for all $f \in S$. We have a morphism

$$\hat{x} : \mathrm{Hom}(-, C) \to M,$$

$$\hat{x}_D : \mathrm{Hom}(D, C) \to M(D),$$

$$f \mapsto M(f)(x).$$

This is actually the natural transformation we obtain from $x$ via the additive Yoneda Lemma. Also note that $I \subseteq \mathrm{Ker}(\hat{x})$. Hence we obtain a morphism

$$\tilde{x} : \ \mathrm{Hom}(-, C)/I \to M.$$

But since $\mathrm{Hom}(-, C)/I \in \mathfrak{T}$, we have that $\tilde{x} = 0$. Hence

$$0 = \tilde{x}_C((1_C) = M(1_C)(x) = x.$$

So $M$ is $\mathcal{J}$-torsion free.

($\Leftarrow$) Suppose the $M$ is $\mathcal{J}$-torsion free. Let $N \in \mathfrak{T}$ and let $\alpha : N \to M$. Let $C \in \mathcal{C}$ and let $x \in N(C)$. So there exists a sieve $S_x \in \mathcal{J}(C)$ with $N(f)(x) = 0$ for all $f \in S_x$. Hence

$$M(f)\alpha_c(x) = \alpha_D(N(f)(x)) = 0.$$

Therefore $\alpha_C(x) = 0$ as $M$ is $\mathcal{J}$-torsion free. It follows that $\alpha = 0$. $\square$

## 2. Prime sieves

**Definition 2.1.** A *pre-filter* $\mathcal{J}$ is an assignment to each object $C \in \mathcal{C}$ a set of sieves $\mathcal{J}(C)$ at $C$ such that for each $C \in \mathcal{C}$ the following condition is satisfied:
If $S$ is a sieve such that $S \in \mathcal{J}(C)$ and $T$ is a sieve such that $S \subseteq T$ then $T \in \mathcal{J}(C)$.

Let $S$ be a sieve at an object $C \in \mathcal{C}$. Define $\mathcal{J}_{C-S}$ to be the following assignment to each object $D \in \mathcal{C}$:

$$\mathcal{J}_{C-S}(D) = \begin{cases} \{T \mid T \nsubseteq S\} & \text{if } D \cong C, \\ \{\mathrm{Hom}(-,C)\} & \text{if } D \ncong C. \end{cases}$$

**Definition 2.2.** Let $\mathcal{J}$ be a pre-filter on $\mathcal{C}$. Let $M$ be a right $\mathcal{C}$-module. Call $M$ *$\mathcal{J}$-torsion free* if whenever there is a $C \in \mathcal{C}$ such that there exists an $x \in M(C)$ along with a sieve $S_x \in \mathcal{J}(C)$ such that $M(f)(x) = 0$ for all $f \in S_x$ then we must have $x = 0$.

**Lemma 2.3.** *Suppose that $\mathcal{C} = R$ is a ring. So sieves correspond to right ideals in $R$. A right ideal $P$ of a ring $R$ is completely prime if and only if $R/P$ is nonzero and $\mathcal{J}_{R-P}$-torsion free.*

**Proof.** ($\Rightarrow$) Suppose that $P$ is completely prime. Let $x \in R$ and $I \in \mathcal{J}_{R-P}$ be such that $(x+P)\cdot I = 0$ in $R/P$. Since $I \nsubseteq P$, there exists $y \in I$ and $y \notin P$. But $yx \in P$, and $P$ is completely prime, hence $x \in P$.

($\Leftarrow$) Suppose that $P$ is $\mathcal{J}_{R-P}$ torsion free. Suppose $xy \in P$ and $x \notin P$. Hence $xR \in \mathcal{J}_{R-P}$ and $(y+P)\cdot xR = 0$. Therefore $y \in P$ as $P$ is $\mathcal{J}_{R-P}$-torsion free. $\square$

**Definition 2.4.** Let $\mathcal{J}$ be a pre-filter on $\mathcal{C}$. Call a right $\mathcal{C}$-module $F$ *weakly $\mathcal{J}$-torsion free* if whenever there exists a $C \in \mathcal{C}$ and an $x \in F(C)$ such that there exists an $S_x \in \mathcal{J}(C)$ with

$$F(gf)(x) = 0,$$

for all $f \in S_x$ and all $g \in \mathrm{Hom}(C,C)$ then $x = 0$.

We call sieve $S$, at $C \in \mathcal{C}$, *w-prime* if the right $\mathcal{C}$-module $\mathrm{Hom}(-,C)/S$ is weakly $\mathcal{J}_{C-S}$-torsion free. We use the notation $C/S$ to denote the right $\mathcal{C}$-module $\mathrm{Hom}(-,C)/S$.

**Lemma 2.5.** *Let $\mathcal{C} = R$ be a ring, not necessarily commutative. So a sieve corresponds to a right ideal in $R$. A right ideal is prime (in the classical sense) if and only if it is w-prime and proper.*

**Proof.** ($\Rightarrow$) Let $P$ be a right prime ideal. Let $I \in \mathcal{J}_{C-P}(C)$ and let $x \in \mathcal{C}$ be such that $(x+P)RI = \{0\}$. So $xRI \subseteq P$. Since $I \in \mathcal{J}_{C-P}$ and since $P$ is prime it follows that $xR \subseteq P$. Hence $x + P = 0$.

($\Leftarrow$) Suppose that $P$ is $w$-prime. Let $IJ \subseteq P$ and $J \nsubseteq P$. So $J \in \mathcal{J}_{C-P}$. So for all $y \in I$ we have that $(y+P)RJ = \{0\}$. Since $P$ is $w$-prime we have that $y \in P$. Hence $I \subseteq P$. $\square$

For any right ideal $I$ in $\mathrm{Hom}(C,C)$ and any arrow $f : D \to C$, we write $If$ for the set $\{hf \mid h \in I\}$, and for any sieve $J$ at $C$, we write $IJ$ for the sieve generated by the set $\bigcup_{f \in J} If$.

**Proposition 2.6.** *A sieve $P$ at $C \in \mathcal{C}$ is w-prime if and only if the following property is satisfied:*

*For any right ideal $I$ in* $\mathrm{Hom}(C,C)$ *and any sieve $J \nsubseteq P$ with $IJ \subseteq P$ then $I \subseteq P$.*

**Proof.** Note that if $P$ is a $w$-prime sieve at $C$ then by definition $C/P$ is weakly $\mathcal{J}_{C-P}$ torsion free, that is if there exists a $D \in \mathcal{C}$ and a morphism $h : D \to C$ and a sieve $J \in \mathcal{J}_{C-P}(D)$ such that $(C/P)(xj)(h) = 0$ for all $x \in \mathrm{Hom}(C,C)$ (that is, $hxj \in P(D')$ for all $x \in \mathrm{Hom}(D,D)$), for any $(j: D' \to D) \in J$ then $h \in P(D)$. This property is automatically satisfied if $D \ncong C$, since $\mathcal{J}_{C-P}(D) = \{\mathrm{Hom}(-,D)\}$.

Thus a sieve $P$ at $C$ is $w$-prime if and only if $P$ satisfies the property that if there exists a morphism $h: C \to C$ and a sieve $J \in \mathcal{J}_{C-P}(C)$(that is $J \nsubseteq P$), such that $hxj \in P(D)$ for all $x \in \mathrm{Hom}(C,C)$ and for any $(j: D \to C) \in J$ then $h \in P(C)$.

($\Rightarrow$) Assume that $P$ is a $w$-prime sieve at $C$. If there exist a right ideal $I$ of $\mathrm{Hom}(C,C)$ and a sieve $J \nsubseteq P$ such that $IJ \subseteq P$ then for any $f \in I$ and for $j: D \to C \in J$ we have that $f \cdot \mathrm{Hom}(C,C) \cdot j \subseteq P(D)$. Hence $f \in P(C)$ by the first paragraph.

($\Leftarrow$) Suppose that $P$ satisfies the above condition. Let $f: C \to C$ and a sieve $J \nsubseteq P$ be such that $f \cdot \mathrm{Hom}(C,C) \cdot j \subseteq P(D)$ for all $(j: D \to C) \in J$, and let $I$ be the sieve at $C$ generated by $f$. Then by definition, $IJ$ is the sieve generated by the set $\{fkj \mid j \in J, k \in \mathrm{Hom}(C,C)\}$, and hence $IJ \subseteq P$. It follows $I \subseteq P$ and hence $f \in P$. $\square$

Now we look at an appropriate categorical version of two-sided ideals.

**Definition 2.7.** A sieve $S$ at $C \in \mathcal{C}$ is called *two-sided* if for any $x : C \to C$ and any $f \in S$ we have $xf \in S$. For each sieve $S$ at $C$, the sieve generated by the set $\{hj \mid j \in S, h \in \mathrm{Hom}(C,C)\}$ is the smallest two-sided sieve containing $S$, denoted by $\bar{S}$.

Let $S$ be a sieve at an object $C \in \mathcal{C}$. Define $\tilde{\mathcal{J}}_{C-S}$ to be the following assignment to each object $D \in \mathcal{C}$:

$$\tilde{\mathcal{J}}_{C-S}(D) = \begin{cases} \{T \mid T \text{contains some two-sided sieve} K \text{and} K \nsubseteq S\} & \text{if} D \cong C, \\ \{\mathrm{Hom}(-,C)\} & \text{if} D \ncong C. \end{cases}$$

By using a similar argument in the proof of Lemma 2.5, we have

**Proposition 2.8.** *A sieve $P$ at $C \in \mathcal{C}$ is w-prime if and only if $C/P$ is $\tilde{\mathcal{J}}_{C-S}$-torsion free.*

In [3], Kelly introduced the notion of an ideal in an additive category as an additive subfunctor of $I \hookrightarrow \mathrm{Hom}(-,-)$. This can be characterised as a set $K$ of arrows such that for all $f, g \in K$, the arrow $u(f+g)v \in K$ for all $u$ and $v$ where this composition makes sense. If $I$ is an ideal we let $I(C)$ denote the set of arrows in $I$ with codomain $C$.

Obviously an ideal $I$ gives us for every object $C \in \mathcal{C}$ a two-sided sieve $I(C)$. The question now arises given a two-sided sieve $S_C$ for each object $C \in \mathcal{C}$, under what conditions can we patch these sieves together to obtain an ideal.

**Lemma 2.9.** *An assignment to each object $C \in \mathcal{C}$ a sieve $K(C)$ is an ideal if and only if the following two properties hold:*

(i) *For each object $C \in \mathcal{C}$, the functor $K(C)$ is a two-sided sieve at $C$.*

(ii) *For each $h : C \to D$ and for any $x \in K(C)$ then $hx \in K(D)$.*

**Definition 2.10.** Given any two sieves $I$ and $J$ at $C$, we define a sieve $IJ$ at $C$, called *the product* of $I$ and $J$, to be the sieve generated by the set $I(C)J$ (as defined above). From Proposition 2.6, we see that a $w$-prime sieve only has a weak version of the primeness with respect to the product. We call a sieve $P$ *prime* if $IJ \subseteq P$ implies either $I \subseteq P$ or $J \subseteq P$.

**Example 2.11.** Let $\mathcal{C}$ be a category with precisely two objects $A$ and $B$. Let $\mathrm{Hom}(A,A) = \{0_A, 1_A\}$, let $\mathrm{Hom}(B,B) = \{0_B, 1_B\}$, let $\mathrm{Hom}(A,B) = \{0_{AB}\}$ and let $\mathrm{Hom}(B,A) = \{0_{BA}, x\}$. There is only one possible composition on this additive category. The only sieves at $A$ are $P = \{0_{BA}, 0_B\}$, the sieve $J = \{x, 0_{BA}, 0_A\}$ and the largest sieve $\mathrm{Hom}(-,A)$. One can check by using Proposition 2.6 that the sieve $P$ is $w$-prime. However, $P$ is not prime since $J^2 \subseteq P$ and $J \nsubseteq P$. This shows that our two notions of primeness are indeed distinct.

**Example 2.12.** Let $\mathcal{C}$ be a category with precisely two objects $A$ and $B$. Let $\mathrm{Hom}(A,A) = Z$, let $\mathrm{Hom}(B,B) = \{0_B, 1_B\}$, let $\mathrm{Hom}(A,B) = \{0_{AB}\}$ and let $\mathrm{Hom}(B,A) = \{0_{BA}, x\}$ (note $2x = 0$). There is only one possible composition on this additive category and note that $2 \circ x = 0$. Let $P = \{0_{BA}, 0_A\}$, $J = \{x, 0_{BA}, 0_A\}$ and $I = \{0_{BA}, 2Z, 0_A\}$. Then $P$, $I$ and $J$ are 2-sided seives at $A$. One can check by using Proposition 2.6 that the sieve $P$ is not w-prime, because neither $J \nsubseteq P$ and $I(A) \nsubseteq P$ but $IJ \subseteq P$. However $P(A) = 0_A$ is prime in the endomorphism ring *Hom*$(A,A)$.

If $A \subseteq \mathrm{Hom}(-,C)$ then we let $\hat{A}$ denote the sieve generated by $A$.

**Lemma 2.13.** (i) If *$I,J$ and $K$ are sieves at $C \in \mathcal{C}$ then $(IJ)K = I(JK)$.*

(ii) If *$I,J$ and $K$ are sieves at $C \in \mathcal{C}$ then $(I+J)K = IK + IK$ and $K(I+J) = KI + KJ$.*

(iii) If *$I,J$ are two-sided sieves at $C \in \mathcal{C}$ then $IJ \subseteq I \cap J$.*

(iv) If *$I,J$ are ideals of* $\mathrm{Hom}(C,C)$ *then $\widehat{I} \cdot \widehat{J} = \widehat{IJ}$.*

**Proof.** Parts (i)–(iii) are obvious.

(iv) From definitions the sieve $\widehat{I} \cdot \widehat{J}$ is generated by the set

$$A = \{ij\alpha \mid j \in J,\ i \in I \text{ and } \mathrm{cod}(\alpha) = C\}.$$

The Lemma follows from the fact that $A \subseteq \widehat{IJ}$. The other inclusion is trivial since $\widehat{I} \cdot \widehat{J} \supseteq IJ$. $\square$

**Lemma 2.14.** *If $P$ is a prime sieve at $C \in \mathcal{C}$ then $P(C)$ is a prime ideal of* $\mathrm{Hom}(C,C)$.

**Proof.** Let $I,J$ be ideals of $\mathrm{Hom}(C,C)$ such that $IJ \subseteq P(C)$. Hence by the preceding lemma and definitions we have

$$P \supseteq \widehat{IJ} = \widehat{I} \cdot \widehat{J} = I \cdot \widehat{J}.$$

Hence if $J \nsubseteq P$ we must have $I \subseteq P$ by Proposition 2.6. □

We are now in the position to show the existance of prime (two-sided) sieves, and hence of $w$-prime sieves. We call a sieve $M$ at $C$ *maximal* if $M \neq \mathrm{Hom}(-C)$ and no proper sieve at $C$ strictly contains it.

**Proposition 2.15.** *Any maximal sieve $M$ at $C \in \mathcal{C}$ is prime.*

**Proof.** Assume $IJ \subseteq M$ with $I \nsubseteq M$. We want to show that $J \subseteq M$. In fact, we have $I + M = \mathrm{Hom}(-,C)$ and hence $I(C) + M(C) = \mathrm{Hom}(C,C)$. Thus there are $z \in I(C)$ and $m \in M(C)$ such that $z + m = 1_C$. Now for each $x : D \to C$ in $J$, we have $x = 1_C x = (z+m)x = zx + mx$ which is in $M$ since $mx \in M$ and $zx \in IJ \subseteq M$. Hence $J \subseteq M$. □

Note that Zorn's Lemma ensures the existance of maximal sieves at any object $C$ of $\mathcal{C}$.

To characterise primeness, in terms of torsion, we have to introduce the following notions.

Let $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Ab}$ be a right $\mathcal{C}$-module. Let $x \in F(C)$, where $C \in \mathcal{C}$. So by the additive Yoneda Lemma, $x$ defines a natural transformation

$$\check{x} : \mathrm{Hom}(-,C) \to F.$$

Call $\mathrm{Im}(\check{x}) \hookrightarrow F$ *the submodule of $F$ generated by $x$*. We use the notation $\widehat{x}$ to denote $\mathrm{Im}\,\check{x}$. More generally let $C_i \in \mathcal{C}$ be a collection of objects in $\mathcal{C}$ with $i \in I$ and $A_i \subseteq F(C_i)$. Set $A = \bigcup_{i \in I} A_i$ and define

$$\widehat{A} = \sum_{x \in A} \mathrm{Im}(\check{x}).$$

We call $\widehat{A}$ *the submodule of $F$ generated by the set $A$*. Now suppose we have sieves $I \hookrightarrow \mathrm{Hom}(-,C)$ and $J \hookrightarrow \mathrm{Hom}(-,C)$. If we let $A = \bigcup_{f \in J} If$ then $\widehat{A}$ is precisely the product $IJ$ as defined earlier.

We now reconcile these ideas with ring theory.

**Example 2.16.** Let $\mathcal{C} = R$ be a ring. Let $M$ be a right $R$-module. Let $x \in M$. Then $\widehat{x}$ is the submodule generated by $x$.

**Proof.** This is clear since

$$\check{x} : R \to M$$

is the canonical $R$-module homomorphism associated to $x$. □

Let $x : D \to C$ in $\mathcal{C}$. Then

$$\widehat{x} = \{xg \mid \operatorname{cod}(g) = D\}.$$

Now let $I \hookrightarrow \operatorname{Hom}(-, C)$ and let $y = [x : D \to C] \in C/I(D)$.

In the above situation, if $A \in \mathcal{C}$ then

$$\widehat{y}(A) = \{[f] \mid f \in \widehat{x} \text{ and } f : A \to C\}.$$

We can characterise the definition of weakly $J$-torsion free in terms of the above notation as follows:

**Lemma 2.17.** *Let $J$ be a prefilter on $\mathcal{C}$. A right $\mathcal{C}$-module $M$ is weakly $J$-torsion free iff the property is satisfied:*

*For any $C \in \mathcal{C}$, any $x \in M(C)$ and any sieve $S_x \in \mathcal{J}(C)$ such that $\widehat{x}(S_x) = \{0\}$ then $x = 0$.*

This inspires the following definition:

**Definition 2.18.** Let $M : \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ be a right $\mathcal{C}$-module. Let $\mathcal{J}$ be a pre-filter on $\mathcal{C}$. The module $M$ is called *strongly $\mathcal{J}$-torsion free at the object $D \in \mathcal{C}$* if whenever there is a $C \in \mathcal{C}$, an $x \in M(C)$ and a $S_x \in \mathcal{J}(D)$ such that $\widehat{x}(f) = 0$ for all $f \in S_x$ then $x = 0$.

**Proposition 2.19.** *Let $P$ be a sieve at $C \in \mathcal{C}$. Then $P$ is prime if and only if $C/P$ is strongly $\mathcal{J}_{C-P}$ torsion free at $C$.*

**Proof.** ($\Rightarrow$) Suppose that $P$ is prime. Let $D \in \mathcal{C}$ and let $y \in (C/P)(D)$. Suppose that $y = [x : D \to C]$. Suppose that $S_y \in \mathcal{J}_{C-P}(C)$ is such that $\widehat{y}(f) = 0$ for all $f \in S_y$.

Note that $\widehat{y}(C) = \{[xg] \mid g \in \operatorname{Hom}(C, D)\}$. Also

$$\begin{aligned} \widehat{y}(f) :\ & \widehat{y}(C) \to \widehat{y}(E) \\ & [xg] \mapsto [xgf]. \end{aligned}$$

Hence $xgf \in P$ for all $g \in \operatorname{Hom}(C, D)$ and for all $f \in S_y$.

Now $\widehat{x} \cdot S_y$ is generated by the set

$$\{xgf \mid g \in \operatorname{Hom}(C, D), f \in S_y\}.$$

Hence $\widehat{x} \cdot S_y \subseteq P$ and since $P$ is prime, we have $x \in P$.

($\Leftarrow$) Suppose that $C/P$ is strongly $\mathcal{J}_{C-P}$ torsion free at $C$. Let $I$ and $J$ be sieves at $C$ with $IJ \subseteq P$. Suppose that $J \nsubseteq P$. Let $(x : D \to C) \in I$. Since $\widehat{x} \hookrightarrow I$ we have that $\operatorname{Im}(\widehat{x}(f)) \subseteq \operatorname{Im}(I(f)) \subseteq P$ for all $f \in J$. Hence $y = [x] = 0$. This implies that $x \in P$. $\square$

## 3. The Separation Lemma: local and global versions

We will give in this section categorical versions of the classical Krull Separation Lemma and of the characterising theorem for semiprime ideals in a noncommutative ring by Nagata and Levitzki.

Recall that a multiplicative set of a ring $R$ is a subset $S$ of $R$ which does not contain zero and for any pair $f$ and $g$ in $S$ the intersection $fRg \cap S \neq \emptyset$. Note that if $R$ is the ring $\mathrm{Hom}(C,C)$, then $fRgR$ is just $(\widehat{f}\cdot\widehat{g})(C)$. As is well known, the classical Separation Lemma ensures the existance of prime ideals of rings. By Examples 2.11 and 2.12, we see also that a two-sided sieve $P$ at $C$ is not necessarily prime even if $P(C)$ is prime and that a prime sieve is not necessarily prime. With a little surprise, we can show that the classical Separation Lemma in fact ensures the existance of prime sieves.

**Definition 3.1.** We call a set $S$ of arrows with codomain $C \in \mathcal{C}$ *a multiplicative set at* $C$ if the following properties are satisfied:
(i) If $f, g \in S$ then $\widehat{f} \cdot \widehat{g}$ intersects $S$.
(ii) For all $D \in \mathcal{C}$, the zero morphism $0_{DC} \in \mathrm{Hom}(D,C)$ is not in $S$.

Now we have a categorical generalisation of the Separation Lemma.

**Theorem 3.2.** *Let $S$ be a multiplicative set at $C \in \mathcal{C}$. Let $I$ be a two-sided sieve at $C \in \mathcal{C}$ which is disjoint with $S$. Then there exists a two-sided prime sieve which contains $I$ and is disjoint with $S$.*

**Proof.** By Zorn's Lemma (recall that $\mathcal{C}$ is small), there is a two-sided sieve $P$ which is maximal with the property that $P \supseteq I$ and $P$ is disjoint with $S$. We now prove that $P$ is prime. Let $J_1 \not\subseteq P$ and $J_2 \not\subseteq P$ with $J_1J_2 \subseteq P$. Then there are $f \in S \cap (J_1 + P)$ and $g \in S \cap (J_2 + P)$, and hence $(\widehat{f} \cdot \widehat{g}) \subseteq (J_1 + P)(J_2 + P) = J_1J_2 + J_1P + PJ_2 + PP \subseteq P$ since $P$ is 2-sided. But we have a contradiction since $(\widehat{f} \cdot \widehat{g}) \cap S$ is not empty and $P$ is disjoint with $S$. $\square$

It is easy to see that a classical multiplicative set of the endomorphism ring $\mathrm{Hom}(C,C)$ is in fact a multiplicative set in our sense. Thus we obtain a stronger version of the Krull Separation Lemma in the following corollary:

**Corollary 3.3.** *Let $S$ be a multiplicative set (in the classical sense) of the ring* $\mathrm{Hom}(C,C)$ *and let $I$ be a two-sided sieve at $C$ with $S \cap I = \emptyset$. Then there is a two-sided prime sieve which contains $I$ and is disjoint with $S$.*

Corollary 3.3 does not say that the sieve $I$ above is an intersection of prime sieves. In fact Example 2.12 gives such a counterexample.

On the other hand, we are able to characterise those sieves which can be expressed as an intersection of two-sided prime sieves. We call such sieves *semiprime*. The following result can certainly be regarded as generalisation of the characterisation theorem of semiprime ideals due to Nagata and Levitzki.

**Theorem 3.4.** *Let $I$ be a two-sided sieve at an object $C \in \mathcal{C}$. Then $I$ is semiprime if and only if whenever $x \in \mathrm{Hom}(-,C)$ with $\widehat{x} \cdot \widehat{x} \subseteq I$ then $x \in I$.*

**Proof.** ($\Rightarrow$) This is clear since each prime sieve has the property described above.

($\Leftarrow$) Now suppose that $I$ has the property stated in the Theorem. We shall show that $I$ is the intersection of all two-sided prime sieves containing $I$. So let $x_0 \in \mathrm{Hom}(-,C)\backslash I$. We have to find a two-sided prime sieve that contains $I$ but not $x_0$.

By the contrapositive of the property stated in the theorem we find that $\widehat{x}_0 \cdot \widehat{x}_0 \nsubseteq I$. So there exists an $x_1 \in \widehat{x}_0 \cdot \widehat{x}_0 \backslash I$. By induction, we obtain a sequence of morphisms $x_0, x_1, x_2, \ldots$ with codomain $C$ and $x_i \notin I$. Also note that $x_{i+1} \in \widehat{x}_i \cdot \widehat{x}_i$.

If $J$ is a two-sided sieve with $x_i \in J$ then $x_{i+1} \in \widehat{x}_i \cdot \widehat{x}_i \subseteq J$. Hence by induction $x_k \in J$ for all $k \geq i$.

Now $I$ is a sieve with the property that $x_i \notin I$ for all $i$. Since the set $S = \{x_i\}$ is a multiplicatively set which is disjoint with $I$, by Theorem 3.2, we have an two-sided prime sieve $P$ which contains $I$ and is disjoint with $S$. In particular, $x_0 \notin P$. $\square$

Recall that in Section 2 we defined an ideal in $\mathcal{C}$ to be a subfunctor of $\mathrm{Hom}(-,-)$. We also gave a characterisation of such ideals as sets of arrows. Now note that the set of all ideals forms a complete lattice under inclusion.

If $K: \hookrightarrow \mathrm{Hom}(-,-)$ is an ideal in $\mathcal{C}$ and $C \in \mathcal{C}$ then we use the notation $K(C)$ to denote the set of all arrows in $K$ with codomain $C$. Note that $K(C)$ is a two-sided sieve at $C$.

If $A$ is a set of arrows in $\mathcal{C}$ we let $\widetilde{A}$ denote the intersection of all ideals containing $A$. Then $\widetilde{A}$ is an ideal which we call *the ideal generated by $A$*. If $A$ happens to be a sieve at an object $C \in \mathcal{C}$ we note that $\widetilde{A}$ is the set of all finite sums of elements in the set

$$\{xi \mid i \in A \text{ and } \mathrm{dom}(x) = C\}.$$

**Lemma 3.5.** *If $I$ is two-sided then $\widetilde{I}(C) = I$.*

**Definition 3.6.** Let $K, L: \hookrightarrow \mathrm{Hom}(-,-)$ be ideals in $\mathcal{C}$. Let $A = \bigcup_{C \in \mathcal{C}} K(C) \cdot L(C)$. We call the ideal $\widetilde{A}$ *the product of $K$ and $L$* and denote it by $KL$.

Similar to the case of product of sieves, one can check that the product is associative.

**Lemma 3.7.**

(i) *If $I, J$ and $K$ are ideals of $\mathcal{C}$ then $(I+J)K = IK + IK$ and $K(I+J) = KI + KJ$.*

(ii) $IJ \subseteq I \cap J$.

**Lemma 3.8.** *If $I, J$ are sieves at $C \in \mathcal{C}$ then $\widetilde{I} \cdot \widetilde{J} = \widetilde{IJ}$.*

**Proof.** Since $\widetilde{I}\widetilde{J} \supseteq IJ$ it is clear that $\widetilde{I}\widetilde{J} \supseteq \widetilde{IJ}$.

To show the reverse inclusion we note that $\widetilde{I}\widetilde{J}$ is the smallest ideal containing $\cup_{D \in \mathcal{C}} \widetilde{I}(D)\widetilde{J}(D)$. The typical element $z$ in $\widetilde{I}(D)\widetilde{J}(D)$ has the form $z = yixj$ where

$j \in J$, $i \in I$ and $x, y \in \mathrm{Hom}(C,D)$ are any morphisms. It follows that $ix \in I$ since $I$ is a sieve. Hence $z \in \tilde{I}J$. □

We call an ideal $K$ *proper* if $K \neq \mathrm{Hom}(-,-)$.

**Definition 3.9.** Let $P$ be a proper ideal in $\mathcal{C}$. We call $P$ *prime* if given any two ideals $K, L$ such that $KL \subseteq P$ then $K \subseteq P$ or $L \subseteq P$. We call an ideal $Q$ *semiprime* if given any ideal $K$ such that $KK \subseteq Q$ then $K \subseteq Q$.

**Proposition 3.10.** *An ideal $K$ is semiprime if and only if $K(C)$ is semiprime for all $C \in \mathcal{C}$.*

**Proof.** ($\Rightarrow$) Suppose $K$ is semiprime and $I^2 \subseteq K(C)$, where $I$ is a sieve at $C \in \mathcal{C}$. Then by Lemma 3.8 $\tilde{I}\tilde{I} = \tilde{I^2} \subseteq K$. Since $K$ is semiprime we have that $\tilde{I} \subseteq K$. Hence $I = \tilde{I}(C) \subseteq K(C)$.

($\Leftarrow$) If $K$ satisfies the property stated in the proposition and $LL \subseteq K$ then for all $C \in \mathcal{C}$

$$L(C)L(C) \subseteq K(C)$$

and so $L(C) \subseteq K(C)$. □

**Definition 3.11.** We call a set $S$ of arrows in $\mathcal{C}$ a *multiplicative set of* $\mathcal{C}$ if the following properties are satisfied:

(i) If $f, g \in S$ then $\widetilde{f} \cdot \widetilde{g}$ intersects $S$.

(ii) For all pairs $D, C \in \mathcal{C}$, the zero morphism $0_{DC} \in \mathrm{Hom}(D,C)$ is not in $S$.

Now we have a global version of the Separation Lemma of which the proof is similar to that of Theorem 3.2.

**Theorem 3.12.** *Let $S$ be a multiplicative set at $C \in \mathcal{C}$. Let $I$ be an ideal which is disjoint with $S$. Then there exists a prime sieve which contains $I$ and is disjoint with $S$.*

**Proof.** By Zorn's Lemma, there is an ideal $P$ which is maximal with the property that $P \supseteq I$ and $P$ is disjoint with $S$. We now prove that $P$ is prime. Let $J_1 \nsubseteq P$ and $J_2 \nsubseteq P$ be two ideals with $J_1J_2 \subseteq P$. Then there are $f \in S \cap (J_1 + P)$ and $g \in S \cap (J_2 + P)$, and hence $(\widetilde{f} \cdot \widetilde{g}) \subseteq (J_1 + P)(J_2 + P) = J_1J_2 + J_1P + PJ_2 + PP \subseteq P$ by Lemma 3.8. But we have a contradiction since $(\widetilde{f} \cdot \widetilde{g}) \cap S$ is not empty and $P$ is disjoint with $S$. □

**Corollary 3.13.** *Each maximal ideal is prime.*

**Theorem 3.14.** *Let $K$ be an ideal in $\mathcal{C}$. Then $K$ is an intersection of primes if and only if $K$ is semiprime.*

**Proof.** ($\Rightarrow$) This is clear since each prime ideal is semiprime.

($\Leftarrow$) Now suppose that $I$ is semiprime. We shall show that $I$ is the intersection of all prime ideals containing $I$. So let $x_0 \notin I$. We have to find a prime ideal that contains $I$ but not $x_0$.

Since $I$ we have that $\widetilde{x}_0 \cdot \widetilde{x}_0 \nsubseteq I$. So there exists an $x_1 \in I \backslash \widetilde{x}_0 \cdot \widetilde{x}_0$. By induction, we obtain a sequence of morphisms $x_0, x_1, x_2, \ldots$ and $x_i \notin I$. Also note that $x_{i+1} \in \widetilde{x}_i \cdot \widetilde{x}_i$.

If $J$ is an ideal with $x_i \in J$ then $x_{i+1} \in \widetilde{x}_i \cdot \widetilde{x}_i \subseteq J$. Hence by induction $x_k \in J$ for all $k \geq i$.

Now $I$ is an ideal with the property that $x_i \notin I$ for all $i$. Since the set $S = \{x_i\}$ is a multiplicative set which is disjoint with $I$, by Theorem 3.11, we have a prime ideal $P$ which contains $I$ and is disjoint with $S$. In particular, $x_0 \notin P$ so that $x_0$ is not in the intersection of all prime ideals containing $I$. $\square$

**Proposition 3.15.** *An ideal $P$ is prime iff it is semiprime and $\cap$-prime.*

Similar to the local case, if $K$ is an ideal of $\mathcal{C}$, then we can define $\mathcal{J}_{\mathcal{C}-K}$ to be the following assignment to each object $D \in \mathcal{C}$:

$$\mathcal{J}_{\mathcal{C}-K}(D) = \{I \hookrightarrow \mathrm{Hom}(-,D) \mid I \text{ contains some two-sided sieve } J \text{ and } J \nsubseteq S(D)\}.$$

Then we have

**Proposition 3.16.** *An ideal $P$ is prime iff $(H(-,-)/P)$ is $\mathcal{J}_{\mathcal{C}-P}$-torsion free.*

In general, $\mathcal{J}_{\mathcal{C}-S}$ is only a filter closed under products.

## 4. The radical

**Definition 4.1.**

(i) Let $R_{\mathrm{R}}$ be the rest of all arrows $y: D \to C$ in $\mathcal{C}$ such that $1_C - yx$ is left invertible for all $x: C \to D$.

(ii) Let $R_{\mathrm{L}}$ be the set of all arrows $y: D \to C$ such that $1_D - xy$ is right invertible for all $x: C \to D$.

The set $R_{\mathrm{L}}$ is the radical of $\mathcal{C}$ as defined in [4, 7].

**Lemma 4.2.** *The sets $R_{\mathrm{R}}$ and $R_{\mathrm{L}}$ are in fact ideals of $\mathcal{C}$.*

**Proof.** We show only that $R_{\mathrm{R}}$ is an ideal. The following argument is due to Street, [7] for the original.

Let $f_1$, $f_2: A \to B \in R_{\mathrm{R}}$. So for each $g: B \to A$ the morphism $1_B - f_1 g$ has a left inverse. Let $h$ be such a left inverse, so that $h - 1_B = h f_1 g$. Now note that

$$\begin{aligned}(1_B - f_2 g h)(1_B - f_1 g) &= 1_B - f_1 g - f_2 g + f_2 g h f_1 g \\ &= 1_B - f_1 g - f_2 g h + f_2 g (h - 1_B) \\ &= 1_B - (f_1 + f_2) g.\end{aligned}$$

It follows that $1_B - (f_1 + f_2)g$ has a left inverse and hence $(f_1 + f_2) \in R_{\mathrm{R}}$.

It is clear that $R_{\mathrm{R}}$ is closed under composition on the right. So let $f: A \to B$ and let $g: B \to X$. It remains to show that $gf \in R_{\mathrm{R}}$. Let $v: X \to A$. We want to show that $1_X - gfv$ has a left inverse. Let $h$ be the left inverse of $1_B - fvg$, so that $hfvg = h - 1_B$. Now we have

$$\begin{aligned}(1_X + ghfv)(1_X - gfv) &= 1_X - gfv + ghfv - ghfvgfv \\ &= 1_X + ghfv - gfv - g(h - 1_B)fv \\ &= 1_X. \qquad \square\end{aligned}$$

By adapting the argument in [7, Proposition 5] we have:

**Theorem 4.3.** *For $y: D \to C$ the following are equivalent:*
(i) $y \in R_{\mathrm{R}}$
(ii) $1_X - zyx$ *is invertible for all* $x: X \to C$ *and* $z: D \to X$.
(iii) $y \in R_{\mathrm{L}}$.

**Proof.** (i)⇒(ii): Let $y \in R_{\mathrm{R}}$. So by the previous lemma $zy \in R_{\mathrm{R}}$. So there exists a $h$ with $h(1_X - zyx) = 1_X$, that is $h = 1_X + hzyx$. Since $-hzyx \in R_{\mathrm{R}}$ we have that $h = 1_X + hzyx$ has a left inverse. Therefore $h$ is invertible and $h^{-1} = (1_X - zyx)$.

(ii)⇒(i): This is clear.

(iii) $\Longleftrightarrow$ (ii): The proof is similar to the above. □

**Corollary 4.4.** *We have $R_{\mathrm{R}} = R_{\mathrm{L}}$*

We call $R_{\mathrm{L}} = R_{\mathrm{R}} = R$ the *radical* of the category $\mathcal{C}$ as defined in [3].

**Definition 4.5.** Let $C \in \mathcal{C}$. We define the *local radical at $C$* to be the intersection of all maximal sieves at $C$. We denote the local radical at $C$ by $R_C$.

**Proposition 4.6.** *Let $R$ be the radical and let $R(C)$ be the set of all arrows in $R$ whose codomain is $C$. Then $R(C)$ is the local radical $R_C$ at $C$ for each object $C \in \mathcal{C}$.*

**Proof.** Let $f: D \to C$ be in $R(C)$ (i.e., $1_C - fg$ is invertible for any $g: C \to D$). Let $M$ be a maximal sieve at $C$ and suppose $f \notin M$. So there exist $k: C \to D$ and $r \in M$ such that $fk + r = 1_C$. Then $r = 1_C - fk$ which is invertible, contradicting $r \in M$.

Conversely, let $f: D \to C \in R_C$. Suppose that there exists a morphism $g: C \to D$ such that $1_C - fg$ is not invertible. So there exists a maximal sieve $M$ with $1_C - fg \in M$. Also $f \in M$ since $f \in R_C$. It follows that $1_C \in M$ contradicting the fact that $M$ is maximal. □

**Corollary 4.7.** *The radical $R = \bigcup_{C \in \mathcal{C}} R_C$. Also in the quotient category we have*

$$\mathcal{C}/R(-, C) = \mathrm{Hom}(-, C)/R_C.$$

**Definition 4.8.** A set of arrows in an additive category $\mathcal{C}$ is called *a right ideal* if it is closed under addition and closed under composition from right hand side.

**Lemma 4.9.** *A set $J$ of arrows is a right ideal iff each $J(C) = \{f \in J \mid \mathrm{cod}(f) = C$ is a sieve at $C\}$.*

**Lemma 4.10.** *A right ideal $M$ is maximal iff there is a unique object $C \in \mathcal{C}$ such that $M(C)$ is a maximal sieve at $C$ and $M(D) = \mathrm{Hom}(-,D)$ for all $D \neq C$.*

**Proposition 4.11.** *The radical $R$ of an additive category is the intersection of all maximal right ideals.*

## 5. The prime spectrum

**Proposition 5.1.** *If $P$ is a prime ideal then $P(C)$ is a prime sieve for each $C \in \mathcal{C}$.*

**Proof.** Let $IJ$ be two sided sieves at $C \in \mathcal{C}$ with $IJ \subseteq P(C)$. Hence $\tilde{I}\tilde{J} = \tilde{IJ} \subseteq P$ and therefore $\tilde{I} \subseteq P$ or $\tilde{J} \subseteq P$. In particular $I = \tilde{I}(C) \subseteq P(C)$ or $J = \tilde{J}(C) \subseteq P(C)$. Therefore $P(C)$ is prime. $\square$

**Example.** Let $\mathcal{C}$ be an additive category consisting of two objects $A$ and $B$. Let $\mathrm{Hom}(A,B) = \{0_{AB}\}$ and let $\mathrm{Hom}(B,A) = \{0_{BA}\}$. An ideal of $\mathcal{C}$ is just the union of an ideal of $\mathrm{Hom}(A,A)$ and an ideal of $\mathrm{Hom}(B,B)$. We take $\mathrm{Hom}(A,A) = \mathrm{Hom}(B,B) = \mathbb{Z}$. Let $P_A$ ($P_B$) be the sieve at $A$ ($B$) with $P_A(A) = 2\mathbb{Z}$ ($P_B(B) = 2\mathbb{Z}$). So $P_A$ and $P_B$ are prime sieves but $P_A \cup P_B = P$ is not a prime ideal of $\mathcal{C}$. To see this consider the sieves $I_A$ and $J_A$ at $A$ with $I_A(A) = 4\mathbb{Z}$ and $J_A(A) = 5\mathbb{Z}$. Similarly we can define $I_B$ and $J_B$. We have $(I_A \cup J_B)(I_B \cup J_A) \subseteq P$, but neither $(I_A \cup J_B) \subseteq P$ nor $(I_B \cup J_A) \subseteq P$.

We let $\mathrm{Spec}(\mathcal{C})$ denote the set of all prime ideals in $\mathcal{C}$. We give $\mathrm{Spec}(\mathcal{C})$ the Zariski topology, that is the topology generated by the open sets

$$D(K) = \{P \in \mathrm{Spec}(\mathcal{C}) \mid P \not\supseteq K\},$$

where $K$ is an ideal of $\mathcal{C}$. If $C \in \mathcal{C}$ then we let $\mathrm{Spec}(C)$ denote the set of two-sided prime sieves at $C$. This set forms a topological space whose typical open set is of the form

$$D(I) = \{P \in \mathrm{Spec}(C) \mid P \not\supseteq I\},$$

$I$ being a sieve at $C$. We can use these ideas to show the relationship between prime sieves and prime ideals.

**Proposition 5.2.** *If $C \in \mathcal{C}$ then the map*

$$\begin{aligned} f_c\colon\ \mathrm{Spec}(\mathcal{C}) &\to \mathrm{Spec}(C) \\ P &\mapsto P(C) \end{aligned}$$

*is continuous.*

**Proof.** The map is well defined by the preceding proposition. One may check that $f_C^{-1}(D(I)) = D(\widetilde{I})$. □

We now show how one may construct a prime ideal given a prime sieve.

**Lemma 5.3.** *Let $K$ and $L$ be ideals of $\mathcal{C}$ then $(K+L)(C) = K(C) + L(C)$.*

**Lemma 5.4.** *Let $P$ be a prime two-sided sieve at an object $C \in \mathcal{C}$. There exists a unique prime ideal $M$ maximal with respect to the property $P = M(C)$.*

**Proof.** Let

$$\mathcal{A} = \{K \hookrightarrow \mathrm{Hom}(-,-) \mid K(C) \subseteq P\}.$$

By Zorn's Lemma $\mathcal{A}$ has a maximal element $M$. We show that $M$ is prime. Let $K$ and $L$ be ideals of $\mathcal{C}$ such that $K \nsubseteq M$ and $L \nsubseteq M$. It follows that $(M+K)(C) \nsubseteq P$ and $(M+L)(C) \nsubseteq P$. Since $P$ is prime we cannot have $(KL)(C) \subseteq P$. Therefore $KL \nsubseteq M$ and so $M$ is prime. It remains to show that $M$ is unique. Let $N$ be another maximal element $A$. Then one can check that $M + N \in \mathcal{A}$ so we must have $N = M + N = M$. □

Given a two-sided prime sieve $P$ at $C \in \mathcal{C}$ we can now associate a prime ideal to $P$ via the preceeding lemma. We denote this prime ideal by $M_P$.

**Theorem 5.5.** *Let $C \in \mathcal{C}$.*

(i) *The map*

$$\begin{aligned} i\colon\ \mathrm{Spec}(C) &\to \mathrm{Spec}(\mathcal{C}) \\ P &\mapsto M_P \end{aligned}$$

*is continuous.*

(ii) *The topological space* Spec($C$) *is a retract of* Spec($\mathcal{C}$), *where the retraction is*

$$f_C\colon \mathrm{Spec}(\mathcal{C}) \to \mathrm{Spec}(C)$$

*as defined in Proposition* 5.2.

**Proof.** (i) One may check that $i^{-1}(D(K)) = D(K(C)$.

(ii) We need to check that $f_C i(P) = i(P)(C) = P$. This is clear since $\widetilde{P} \in \mathcal{A}$, where $\mathcal{A}$ is defined as in the proof of the previous lemma. □

By a similar argument one can show for each $C \in \mathcal{C}$ that Spec(Hom$(C, C)$) is a retrect of Spec$(C)$.

## 6. Artinianness and semisimplicity

In the next two sections we follow proofs given in [2, 3].

**Definition 6.1.** Let $C \in \mathcal{C}$. We call $C$ a *right Artinian object* (*or right Noetherian object*) if any descending (or ascending) chain of sieves at $C$ is finite.

A sieve $I$ at $C \in \mathcal{C}$ is called nilpotent if there exists a positive integer $n$ such that $I^n = \{0\}$.

We now give a categorical version of a theorem due to Brauer.

**Theorem 6.2.** *Let $C \in \mathcal{C}$ be a right Artinian object. If $I$ is a nonnilpotent sieve at $C$ then $I(C)$ contains a nonzero idempotent arrow.*

**Proof.** By Zorn's Lemma and since $C$ is right Artinian the set of all nonnilpotent sieves at $C$ contained in $I$ has a minimal member. Let $I_1$ be such a minimal member.

By the minimality of $I_1$ we must have $I_1^2 = I_1$. So the set of all nonnilpotent sieves $J$ at $C$ satisfying $JI_1 \neq \{0\}$ is nonempty. By applying the right Artinian property of $C$ we find that this set has a minimal member $J_1$. Since $J_1 I_1 \neq \{0\}$, there exists an arrow $u \in J_1(C)$ such that $uI_1 \neq \{0\}$.

By the minimality of $J_1$ we must have $uI_1 = J_1$. So there exists an arrow $a \in I_1(C)$ such that $ua = u$. It follows that $u = ua^n$ for each positive integer $n$.

Let $A = \{f \in I_1 \mid uf = 0\}$. Now $A$ is a sieve at $C$ strictly contained in $I_1$. Hence by the minimality of $I_1$ the sieve $A$ is nilpotent. By the previous paragraph the arrow $a^2 - a$ lies inside $A$. Since $A$ is nilpotent there exists a positive integer $l$ such that $(a^2 - a)^l = 0$. By expanding this we find that $a^l = a^{l+1}g(a)$ for some polynomial $g(x) \in \mathbb{Z}[X]$. By induction we have that $a^l = a^{2l}g(a)^l$. So there arrow $e = a^l g(a)^l$ is idempotent and clearly lies inside $I$. If $e$ were 0 then $a$ would be nilpotent implying that $u$ would be 0. This contradicts the choice of $u$. So $e$ is nonzero. □

**Corollary 6.3.** *Let $C \in \mathcal{C}$ be a right Artinian object. A sieve $I$ at $C$ is nilpotent if and only if $I(C)$ consists of nilpotent arrows.*

**Lemma 6.4.** *If $C \in \mathcal{C}$ is an object with the property that every nonzero two-sided sieve at $C$ not nilpotent then every nonzero sieve at $C$ is not nilpotent.*

**Proof.** Let $I$ be a nilpotent sieve at $C$. Let

$$J = \{xy \mid x \in \mathrm{Hom}(C, C) y \in I\} = \mathrm{Hom}(C, C)I,$$

so that $J$ is a two-sided sieve. Now observe that $J^2$ is generated by the set $\{xyv \mid x \in \mathrm{Hom}(C,C), y \in I(C), v \in I\}$. It follows that $J^2 = \mathrm{Hom}(C,C)I^2$. By induction we find that $J^n = \mathrm{Hom}(C,C)I^n$. Since $I$ is nilpotent there exists an integer $l$ such that $I^l = \{0\}$. Hence $J^l = \{0\}$. But since $J$ is two-sided we must have $J = \{0\}$ and hence $I = \{0\}$. $\square$

We call an object $C \in \mathcal{C}$ *nil-semisimple* if the only nilpotent two-sided sieve at $C$ is the zero sieve.

**Theorem 6.5.** *Let $C \in \mathcal{C}$ be a nil-semisimple right Artinian object. Any sieve at $C$ can be generated by an idempotent of the ring* $\mathrm{Hom}(C,C)$.

**Proof.** Let $I \hookrightarrow \mathrm{Hom}(C,C)$ be a sieve at $C$. If $I$ is the zero sieve then $I$ is generated by $O_C \in \mathrm{Hom}(C,C)$. So assume now that $I$ is nonzero and hence is not nilpotent. By Theorem 6.2, $I(C)$ contains nonzero idempotent arrows. For each such nonzero idempotent $e$ define

$$A(e) = \{x \in I \mid ex = 0\}.$$

So $A(e)$ is a sieve at $C$. Choose $e \neq 0$ such that $A(e)$ is minimal.

If $A(e) \neq \{0\}$ then this sieve contains an idempotent $e_1$. So the arrow $e_2 = e_1 + e - e_1 e$ is an idempotent of $I$. Since $ee_2 = e^2 = e$, we have $A(e_2) \subseteq A(e)$. This inclusion is strict as $ee_1 =$ and $e_2 e_1 = e_1 \neq 0$. It also that $e_2$ is nonzero and hence contradicts the minimality of $A(e_1)$.

Therefore we must have $A(e) = \{0\}$. Since $x - ex \in A(e)$ for all $x \in I$ we must have that $ex = x$ for all $x \in I$. The statement of the theorem now follows since $I = \widehat{e}$. $\square$

**Definition 6.6.** We call an additive category $\mathcal{C}$ *right Artinian* if each object in $\mathcal{C}$ is right Artinian.

**Example 6.7.** Let $\mathcal{C}$ be a category (additive) with objects $C_1, C_2, C_3, \ldots$, that is with countably infinite objects. Let $\mathrm{Hom}(C_i, C_j) = \{0\}$ for $i \neq j$ and $\mathrm{Hom}(C_i, C_i) = \mathbb{Z}_2$. So a sieve at an object $C$ is just an ideal of the ring $\mathrm{Hom}(C,C)$. For each $i \in \mathbb{N}$ we define an descending chain of sieves:

$$I_{1i} \supseteq I_{2i} \supseteq I_{3i} \supseteq \cdots$$

by $I_{ij} = \mathbb{Z}_2$ for $i \leq j$ and $I_{ij} = \{0\}$ for $i > j$. For each $i \in \mathbb{N}$ define an ideal $K_i$ of $\mathcal{C}$ to be the ideal generated by $\cup_{j \in \mathbb{N}} I_{ij}$. So we have an infinite descending chain of ideals

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots,$$

where each inclusion is strict. However the category $\mathcal{C}$ is right Artinian.

**Example 6.8.** As there are rings (noncommutative) that are Artinian with respect to ideals but not right Artinian we have examples of additive categories that are Artinian but not right Artinian.

We call $\mathcal{C}$ *nilsemisimple* if the only nilpotent ideal of $\mathcal{C}$ is the zero ideal.

**Lemma 6.9.** *The additive category $\mathcal{C}$ is nilsemisimple if and only if each object in $\mathcal{C}$ is nilsemisimple.*

**Proof.** ($\Rightarrow$) Suppose that $\mathcal{C}$ is nilsemisimple. Let $C \in \mathcal{C}$ and let $I \hookrightarrow \mathrm{Hom}(-,C)$ be a sieve at $C$. Suppose that $I$ is two-sided and $I^n = \{0\}$. Then

$$(\widetilde{I})^n = \widetilde{I^n} = 0_I.$$

Hence $I = \widetilde{I}(C) = 0_S$.

Note $0_I$ and $0_S$ denote the zero ideal and the zero sieve respectively.

($\Leftarrow$) Suppose each object in $\mathcal{C}$ is nilsemisimple. Let $K \hookrightarrow \mathrm{Hom}(-,-)$ be an ideal such that $K^n = \{0\}$. Hence $K(C)^n = 0_S$ for all $C \in \mathcal{C}$ which implies $K(C) = 0_S$ for all $C \in \mathcal{C}$. Since $K$ is generated by the $K(-,C)$ we have that $K = 0_I$. $\square$

**Theorem 6.10.** *Let $\mathcal{C}$ be a right Artinian nilsemisimple category. Then every ideal of $\mathcal{C}$ is generated by a family of idempotents $(e_C)_{C\in\mathcal{C}}$ where $e_C \in \mathrm{Hom}(C,C)$.*

**Proof.** Let $K \hookrightarrow \mathrm{Hom}(-,-)$ be an ideal. The ideal $K$ is generated by the sieves $K(C)$ which are in turn generated by idempotents by Theorem 6.5. $\square$

**Lemma 6.11.** *Let $C \in \mathcal{C}$. If $R_C$ is the zero sieve then $C$ is nilsemisimple.*

**Definition 6.12.** We call a right $\mathcal{C}$-module $M$ *simple* if $M$ is not the zero module and the only submodules of $M$ are the zero module and the module $M$.

**Definition 6.13.** We call a right $\mathcal{C}$-module $M$ *semisimple* if for every submodule $N \hookrightarrow M$ of $M$ there is a submodule $P \hookrightarrow M$ of $M$ with $M = N \oplus P$.

Following the proofs in [4], we obtain the following results.

**Theorem 6.14.** *If $M$ is a right $\mathcal{C}$-module then the following are equivalent:*
(i) *$M$ is semisimple.*
(ii) *$M$ is the direct sum of its simple submodules.*
(iii) *$M$ is the sum of its simple submodules.*

**Corollary 6.15.** *If $C \in \mathcal{C}$ and $\mathrm{Hom}(-,C)$ is semisimple then $\mathrm{Hom}(-,C)$ is nilsemisimple.*

If the functor $\mathrm{Hom}(-,C)$ is semisimple then we call the object $C \in \mathcal{C}$ *semisimple.* We call an object $C \in \mathcal{C}$ *semisimple* if $\mathrm{Hom}(-,C)$ is a semisimple right $\mathcal{C}$-module.

**Theorem 6.16.** *The following are equivalent:*
(i) *All short exact sequences in* **mod**-$\mathcal{C}$ *split.*
(ii) *All right* $\mathcal{C}$*-modules are semisimple.*
(iii) *All finitely generated right* $\mathcal{C}$*-modules are semisimple.*
(iv) *All cyclic right* $\mathcal{C}$*-modules are semisimple.*
(v) *All objects in* $\mathcal{C}$ *are semisimple.*

## 7. The Wedderburn–Artin Theorem

In [7] the categorical Wedderburn–Artin Theorem was proved up to Morita-equivalence. Here we give a proof more closely related to the original. We shall also give a categorical version of the result that Artinian implies Noetherian.

**Proposition 7.1.** *The following conditions are equivalent for any* $C \in \mathcal{C}$*:*
(i) $C$ *is right Artinian and* $R_C = \{0\}$*;*
(ii) $C$ *is right Artinian and nilsemisimple;*
(iii) $C$ *is semisimple.*

**Proof.** (i) implies (ii) by Lemma 6.11. Also (ii) implies (iii) since by Theorem 6.5 each sieve $I$ can be generated by an idempotent $e$ and hence $\mathrm{Hom}(-,C) = I \oplus (\widehat{1-e})$.
(iii) $\Rightarrow$ (i): Suppose that $\mathrm{Hom}(-,C) = I_1 \oplus I_2 \ldots \oplus I_n$. Let $J$ be a sieve at $C$ then

$$J = \mathrm{Hom}(-,C)\cdot J = I_1J + I_2J + \cdots + I_nJ.$$

Since each $I_j$ is minimal we must have $I_jJ = \{0\}$ or $I_jJ = I_j$. It follows now that there are only finitely many sieves at $C$. $\square$

**Corollary 7.2.** *Every prime sieve in a right Artinian category with zero radical is maximal.*

By Corollary (4.5) we have the following:

**Lemma 7.3.** *If* $\mathcal{C}$ *is right artinian, then for each* $C \in \mathcal{C}$ $C/R_C$ *is a finite direct sum of simple sieves at* $C$ *and hence is both Artinian and Noetherian.*

Let $C \in \mathcal{C}$ and let $I \hookrightarrow \mathrm{Hom}(-,C)$ be a sieve. We call $I$ minimal if $I$ is not the zero sieve and the only sieve properly contained within $I$ is the zero sieve. Observe that all minimal sieves are simple when regarded as right $\mathcal{C}$-modules, that is the only proper submodule of a minimal sieve is the zero sieve.

If $I$ is a minimal sieve at $C$ we define $B_I$ to be the sum of all minimal sieves at $C$ that are isomorphic to $C$ as right $\mathcal{C}$-modules.

**Lemma 7.4.** *In the above situation:*
(i) $B_I$ *is a two-sided sieve at* $C$*.*
(ii) *If* $I$ *and* $J$ *are minimal sieves at* $C$ *that are not isomorphic then* $B_I \cdot B_J = \{0\}$*.*

**Proof.** (i) Let $J$ be a minimal sieve at $C$ such that $J \cong I$. Let $x \in \mathrm{Hom}(C, C)$. The sieve $x \cdot J = \{xj \mid j \in J\}$ is the image of $J$ and since $J$ is a simple right $C$-module we have that $xJ = \{0\}$ or $xJ \cong J$. In either case we have $xJ \subseteq B_I$.

(ii) It suffices to show that $IJ = \{0\}$. Suppose there exists $i \in I(C)$ such that $iJ \neq \{0\}$, so $iJ = I$. We have a contradiction as $J \cong iJ = I$. □

We call an object $C \in \mathcal{C}$ *simple* if the only two-sided sieves at $C$ are $\mathrm{Hom}(-, C)$ and the zero sieve.

**Theorem 7.5.** *If $C \in \mathcal{C}$ is simple then the following are equivalent:*
(i) *$C$ is right Artinian.*
(ii) *$C$ is semisimple.*
(iii) *There is a minimal sieve at $C$.*

**Proof.** Observe that a simple object is nilsemisimple and hence (i) and (ii) are equivalent by the previous lemma's. Also (i) implies (iii) trivially. Now suppose that (iii) holds. Let $I$ be the minimal sieve at $C$. Since $C$ is simple, we have that $B_I = \mathrm{Hom}(-, C)$. So (ii) follows. □

**Lemma 7.6.** *If $C$ is an Artinian object of an additive category, then there is a positive integer $n$ such that $R_C^n = 0$.*

**Proof.** Suppose that $R_C$ is not a nilpotent sieve at $C$. By Theorem 6.1, $R_c$ contains a nonzero idempotent arrow, say $e$. Since $1_C - e$, is not invertible, there is a maximal sieve $M$ containing $1_C - e$, which does not contain $e$, which is a contradiction. □

**Corollary 7.7.** *If $C$ is an Artinian object of an additive category, then $R_C$ is the intersection of all prime sieves at $C$.*

By Lemma 7.1 and Corollary 7.7, we have

**Lemma 7.8.** *If $C$ is an Artinian object then each prime sieve is a maximal sieve.*

Hence if $C$ is an Artinian object of an additive category, then each prime two-sided sieve at $C$ is two-sided maximal.

Now we are going to establish a categorical version of the classical result that each right Artinian ring is right Noetherain. First we have the following lemma.

Since the five Lemma is valid for Abelian categories, we can apply the classical argument to obtain:

**Lemma 7.9.** *If $S$ is a submodule of $F$ then $S$ and $F/S$ are Artinian (or Noetherian) if and only if $F$ is Artinian (or Noetherian).* □

**Theorem 7.10.** *If $C$ is right Artinian then $\mathcal{C}$ is right Noetherian.*

**Proof.** By Proposition 7.1 and Lemma 7.9, it suffices to show that $R_C$ is Noetherian for each $C \in \mathcal{C}$. By Lemma 7.6, it suffices to show that each $R^i_C/R^{i+1}_C$ satisfies ACC for each $C \in \mathcal{C}$. One may check that $R^i_C/R^{i+1}_C$ is a right $\mathcal{C}/R$-module. Then by Theorem 6.16 and Proposition 7.1 $R^i_C/R^{i+1}_C$ is a direct sum of simple $\mathcal{C}/R$-modules. On the other hand $R^i_C/R^{i+1}_C$ is right Artinian by Lemma 7.9 so it must be a finite direct sum of simple $\mathcal{C}/R$-modules and hence satisfies ACC. □

To establish an categorical version of the Wedderburn–Artin theorem, we have to introduce the following:

**Definition 7.11.** An additive category $\mathcal{C}$ is called *normed* if to each object $a \in \mathcal{C}$, one may associate a natural number $|a|$.

A *matrix category over a normed additive category* $\mathcal{C}$, denoted by **Mat($\mathcal{C}$)**, is an additive category whose objects are the objects in $\mathcal{C}$; morphisms from $a$ to $b$ are $|a| \times |b|$-matrices over $\mathrm{Hom}(a,b)$. The composition is the usual matrix product.

It is clear that for any normed additive category $\mathcal{C}$, $\mathcal{C}$ and Mat($\mathcal{C}$) are Morita equivalent.

**Example.** (1) Each $n \times n$-matrix ring $\mathrm{Mat}_n(K)$ over a ring $K$, is a marix category over a normed one-object additive category, where $| * | = n$.

(2) If objects in $C$ are natural numbers and each $\mathrm{Hom}\, n, m$ is a fixed ring $K$, then to each object $n$ associates the same number $n$, we see that Mat($A$) is simply the classical matrix category $\mathbf{Mat}_K$ over a ring $K$.

The classical Wedderburn–Artin theorem says that a semisimple Artinian ring is canonical isomorphic to a finite product of matrix rings over division rings.

Following Street, the local product of a family categories $C_i$ which have the same objects, is the category whose objects are the same as in $C_i$ and whose hom groups $C(A,B)$ given by the product of all hom groups $C_i(A,B)$ and composition is componentwise. The Wedderburn–Artin theorem just says that a semisimple artinian ring is isomorphic to a finite local product of matrix category over a division ringoid.

Given an additive category $\mathcal{C}$, let $B_0$ be the full subcategory of $\mathcal{C}$-Mod consisting of all simple modules, and $B_1$ the full subcategory of $\mathcal{C}$-Mod consisting of all finite direct sums of simple modules.

**Proposition 7.12.** (i) $B_0$ *is a division ringoid.*

(ii) $B_1$ *is equivalent to a local product of matrix categories over normed division ringoid, where for each* $X \in B_1$, $|X|$ *is the number of equivalence class of nonisomorphic simple submodules of* $X$.

**Proof.** For each $X \in B_1$, we have $X = n_1X_1 \oplus n_2X_2 \cdots n_mX_m$, where each $X_i$ is a simple submodule of $X$ and $nX_i$ denotes the coproduct of $n$-copies of $X_i$, and $X_i \ncong X_j$ for any $i \neq j$.

Thus, for each $X$, we have a sequence $\{X_1, X_2, \dots X_n, \dots\}$ of simple submodule of $X$, where $X_n = 0$ if $n \geq m_X$.

For each $\alpha \in \Pi_{X \in C} X_i$, let $n_{\alpha_X}$ denote the number of isomorphic copies of $\alpha_X$, and define $C_\alpha$ to be the (nonfull) subcategory of $C$ with the same objects as in $C$, and hom-group $C_\alpha(X, Y) = C(n_{\alpha_X}\alpha_X, n_{\alpha_Y}\alpha_Y$. Thus, we claim that each $C_\alpha$ is a matrix category over a normed division ringoid, where $|X| = n_\alpha$:

In fact, we see that $C(X_i, Y_j)$ is a division ring (it may be zero) for all $i, j$, and that $C(n_i(X)X_i, n_j(Y)Y_j)$ is the Abelian group of all $n_i(X) \times n_i(Y)$-matrices over $C(X_i, Y_j)$.

By the fact that any morphism between two nonisomorphic simple modules is zero, we see that morphisms in $C(X, Y)$ bijectively corresponds to a family $n_i(X)X_i \to n_j(Y)Y_j$, which corresponds to an element of the product $\Pi_\alpha C(n_{\alpha_X}\alpha_X, n_{\alpha_Y}\alpha_Y)$, the conclusion follows. $\square$

For each object $X$ in a semisimple category $C$, there is a finite decomposition $\mathrm{Hom}(-, X) = n_1X_1 \oplus n_2X_2 \cdots n_mX_m$, where each $X_i$ is a simple submodule of $\mathrm{Hom}(-, X)$ and $nX$ denotes the coproduct of $n$-copies of $X$, and $X_i \ncong X_j$ for any $i \neq j$. By proposition above, we have the following categorical version of Wedderburn–Artin theorem.

**Theorem 7.13.** *A semisimple category $C$ is equivalent to a local product of matrix categories over division ringoids.*

**Corollary 7.14.** *If $C$ is a semisimple category with finitely many objects, then $C$ is equivalent to a finite local product of matrix categories over division ringoids.*

We can add one more equivalent condition to Theorem 7.5.

**Proposition 7.15.** *If $C$ is simple, then the following are equivalent:*

(i) *$C$ is Artinian;*

(ii) *Each* $\mathrm{Hom}(-, X)$ *has a minimal submodule;*

(iii) *$C$ is semisimple;*

(iv) *$C$ is equivalent to a matrix category over a division ringoid.*

**Proof.** Assume (i) and (iii). We want to show (iv). Let $C_\alpha$ be as the one in the proof of Proposition 7.12. If there is some nontrivial decomposition $\mathrm{Hom}(-, X) \cong X_1 \oplus X_2 \cdots$, say $X_i \ncong X$ for some $X \in A$, let $\alpha$ be any element in $\prod X_i$ with $\alpha_X = X_i$. Then $C_\alpha$ is canonically embedded as a not-full subcategory of $C$, and hence the set of all morphisms in $C_\alpha$ can be generated as a proper ideal of $C$ — a contradiction. Thus each $X \in C$ is isomorphic to a finitely many coproduct of a simple module; or equivalently, $C$ is a matrix category over a division ringoid.

The converse is also true. Note that each object in a division ringoid is simple. A matrix category over a division ringoid is canonically equivalent to the category consisting of those objects $a^{n_a}$, which is semisimple since a finite direct sum of simple modules is semisimple. $\square$

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